



Quark determinant in domain-like gluon fields

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We address the computation of the determinant for the Dirac operator corresponding to a quark propagating in a background gluon field of the following type: the gauge field is covariantly constant and self-dual inside a hypersphere and with quark fields satisfying bag-like (chiral-violating) boundary conditions at the surface of the hypersphere. We find that the parity odd part of the logarithm of the determinant corresponds to the Abelian anomaly. However the parity even part also depends on the chiral angle associated with the boundary. This unexpected dependence is discussed.

1 Introduction

The profound physical phenomena expected to arise from QCD, such as confinement, spontaneous chiral symmetry breaking and consequences of the Abelian anomaly, have inspired a range of simplified mathematical settings for the QCD equations and searches for exact solutions to such problems. The exact self-dual and finite action solutions to the Yang-Mills classical equations known as the instanton are one example of this. In the quark sector, a similar endeavour has developed involving searches for exact solutions to the Dirac eigenvalue spectrum for given nontrivial background gluon fields, and in turn computations of the quark determinant (and thus the quark free energy) in such gluonic backgrounds. Often using zeta function regularisation or heat-kernel methods, such problems have now been solved for various compact spaces in various dimensions with and without nontrivial gluon fields in the background. In this paper we shall address such a computation for the quark determinant in what we have previously called a “domain-like” gluon background field, based on our earlier solution to the Dirac eigenvalue problem for such a configuration.

The elements of the problem are as follows: we consider a four-dimensional Euclidean hypersphere of radius R . On the surface, fermion fields satisfy a chirality violating boundary condition reminiscent of bag models of the nucleon, but now for four-dimensional domains with nontrivial field internal to the domain. In various space-time dimensionalities, recent works have addressed such boundary conditions [1, 2, 3, 4, 5, 6, 7] illustrating the relevance of the problem to the realisation of chiral symmetries in QCD. For example, in [1] the fermion boundary condition is regarded

as a mechanism for dynamical generation of the CP-breaking θ -term.

An interesting feature of these boundary conditions is that they render the spectrum of the Dirac operator asymmetric under $\lambda \rightarrow -\lambda$ which in turn has fascinating consequences. In particular, a careful analysis of the fermion determinant leads to an additional asymmetry spectral function [8, 9, 10], as well as the usual zeta function type contributions. In this respect, the four dimensional theory we tackle resembles three dimensional field theories where parity violating effects are well-known and accessed through such asymmetries [8]. The new element we introduce here is the presence inside the hyperspherical region of a non-vanishing covariantly constant, self-dual gluon field.

This type of gluon background field arises in the “domain model” ansatz for the QCD vacuum [11], consisting of an ensemble of such hyperspheres with randomly assigned field orientations and self-duality or anti-self-duality. The fermion boundary condition in this case arises as a result of requiring finite action fluctuations in the presence of a singular field defining the boundary. A chiral angle associated with fermion boundary condition here becomes an additional random variable associated with a given domain, such that averaging over the angle for an ensemble of domains restores chiral symmetry for the ensemble [12]. In such a model, we have previously demonstrated confinement of static quarks [11] and have shown that the chirality of low lying Dirac modes exhibit strong correlation with the duality of the gluon field inside the domain [12] reminiscent of lattice QCD calculations [13, 14, 15] and consistent with the dominance of the chiral properties of the lowest modes in condensate formation, as expressed in the well-known Banks-Casher relation [16]. The computation of the quark determi-

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nant, tackled in this paper, is crucial for studying chiral symmetry realisation in the domain model, and can be important for understanding this problem in QCD itself. In particular, the determinant is precisely the relevant quantity for obtaining the Abelian anomaly in the path-integral approach [17]. The calculation presented here recovers the Abelian anomaly, which is identified as the parity odd part of the logarithm of determinant. The anomaly gets contributions from both the standard ζ -function of the squared Dirac operator and the spectral asymmetry function, since in the presence of chirality violating boundary conditions the spectral ζ -function and spectral asymmetry function do not have direct physical meaning as parity even and odd respectively. The dependence of the anomaly on the chiral angle turns out to be multivalued, which is to be expected since the topological charge here is not restricted to integer values. The parity even part also shows a dependence on the chiral angle (the parameter of boundary conditions), which we consider to be more an artifact of the incompleteness of our calculation than an established property of the determinant. Here we will only identify those potential contributions which are beyond the scope of the present calculation, and which can eliminate the chiral angle dependence of the parity even part.

2 Boundary Conditions, Dirac operator and eigenvalues

Our starting point is the eigenvalue problem for the Dirac operator subject to the bag-like boundary condition

$$D\psi(x) = \lambda\psi(x), \quad (1)$$

$$\not{n}(x)\gamma_5 e^{\vartheta\gamma_5}\psi(x) = \psi(x), \quad x^2 = R^2. \quad (2)$$

Here $\eta_\mu(x) = x_\mu/|x|$, D_μ is the covariant derivative in the fundamental representation,

$$\begin{aligned} D_\mu &= \partial_\mu - i\hat{B}_\mu \\ &= \partial_\mu + \frac{i}{2}\hat{n}B_{\mu\nu}x_\nu, \\ \hat{n} &= t^a n^a, \\ \hat{B}_{\mu\nu} &= \pm B_{\mu\nu}, \end{aligned}$$

where the (anti-)self-dual tensor $B_{\mu\nu}$ is constant, and the Euclidean γ -matrices are in an anti-hermitean representation. To be specific, we will take the field as being self-dual. The magnitude of the topological charge per domain will be a useful quantity for later, and here it turns out to be [11] just $q = B^2 R^4/16$. As in [1], eigenvalues defined by Eqs. (1) and (2) are

real for all real ϑ . In this case the boundary condition for $\bar{\psi}$ required by the condition

$$\bar{\psi}(x)\not{n}(x)\psi(x) = 0, \quad x^2 = R^2,$$

is given by hermitian conjugating Eq. (1). Complex values of the chiral angle require in general construction of a bi-orthogonal basis. Crucial to the solution derived in [12] was the decomposition

$$\psi = i\not{n}\chi + \varphi, \quad \bar{\psi} = i\bar{\chi}\not{n} + \bar{\varphi}, \quad (3)$$

φ and χ having the same chirality. Solutions are then labelled via the Casimirs and eigenvalues

$$\begin{aligned} \mathbf{K}_1^2 &= \mathbf{K}_2^2 \rightarrow \frac{k}{2}\left(\frac{k}{2} + 1\right), \quad k = 0, 1, \dots, \infty \\ K_{1,2}^z &\rightarrow m_{1,2}, \\ m_{1,2} &= -k/2, -k/2 + 1, \dots, k/2 - 1, k/2, \\ k &= 0, 1, 2, \dots, \end{aligned}$$

corresponding to the angular momentum operators $\mathbf{K}_{1,2} = \frac{1}{2}(\mathbf{L} \pm \mathbf{M})$ with \mathbf{L} the usual three-dimensional angular momentum operator and \mathbf{M} the Euclidean version of the boost operator. Thus the solutions for the self-dual background field can be written

$$\psi_{km_1}^{-\kappa} = i\not{n}\chi_{km_1}^{-\kappa} + \varphi_{km_1}^{-\kappa}. \quad (4)$$

The minus sign label on these solutions refers to their negative chirality, $\gamma_5\varphi = -\varphi$, $\gamma_5\chi = -\chi$, and is the only choice of chirality for which the boundary condition Eq. (2) can be implemented for the self-dual background field [12]. (For an anti-self-dual field, the chiralities become positive.) The label $\kappa = \pm$ refers to the polarisation of the spinors according to the combined colour-spin projection specified by the projector

$$O_\kappa = N_+\Sigma_\kappa + N_-\Sigma_{-\kappa} \quad (5)$$

with

$$\begin{aligned} N_\pm &= \frac{1}{2}(1 \pm \hat{n}/|\hat{n}|), \\ \Sigma_\pm &= \frac{1}{2}(1 \pm \boldsymbol{\Sigma} \cdot \mathbf{B}/B) \end{aligned}$$

being respectively separate projectors for colour and spin polarizations.

In the ζ -regularised determinant an arbitrary scale μ appears and it is convenient to work with scaled variables

$$\beta = \sqrt{2\hat{B}}/\mu, \quad \rho = \mu R, \quad \xi = \lambda(B)/\mu, \quad \xi_0 = \lambda(0)/\mu, \quad (6)$$

and where the dimensionless quantity $z = BR^2/2 = \beta^2 \rho^2/4$ appears prominently. Then, as derived in [12], the equations determining the (rescaled) eigenvalues ξ for the two polarisations that appear in the problem are

$$\begin{aligned} M(-\xi^2/\beta^2, k+1, z) \\ + e^\vartheta \frac{\xi \rho}{2(k+1)} M(1 - \xi^2/\beta^2, k+2, z) = 0 \end{aligned} \quad (7)$$

for $\kappa = -$ and

$$\begin{aligned} M(k+2 - \xi^2/\beta^2, k+2, z) \\ + e^\vartheta \frac{\beta^2 \rho}{2\xi} \left[M(k+2 - \xi^2/\beta^2, k+2, z) \right. \\ \left. - \frac{k+2 - \xi^2/\beta^2}{k+2} M(k+3 - \xi^2/\beta^2, k+3, z) \right] = 0 \end{aligned} \quad (8)$$

for $\kappa = +$. The confluent hypergeometric, or Kummer, functions $M(a, b, z)$ emerged originally as solutions of the radial equation for a scalar field in the presence of the above covariant constant gauge field on a finite hypersphere [11].

3 Computing the determinant

We now consider the quark determinant in a single self-dual domain of volume $v = \pi^2 R^4/2$

$$\begin{aligned} \det_\vartheta \left(\frac{i \mathcal{D}}{i \not{\partial}} \right) &= \exp \sum_{k,n,m_1} \left(\ln \frac{\lambda_{kn}^\kappa(B)}{\mu} - \ln \frac{\lambda_{kn}^\kappa(0)}{\mu} \right) \\ &= \exp \{ -\zeta'(s) \}_{s=0} \end{aligned} \quad (9)$$

where the arbitrary scale μ plays a role similar to the renormalization point in other regularisation prescriptions. The zeta function $\zeta(s)$ breaks up into two parts [8] respectively symmetric (S) and antisymmetric (AS) with respect to $\lambda \rightarrow -\lambda$

$$\zeta(s) = \zeta_S(s) + \zeta_{AS}(s), \quad (10)$$

with

$$\begin{aligned} \zeta_S(s) &= \frac{1}{2} (1 + e^{\mp i\pi s}) \zeta_{\mathcal{P}^2}(s/2), \\ \zeta_{AS}(s) &= \frac{1}{2} (1 - e^{\mp i\pi s}) \eta(s) \end{aligned}$$

and the key quantities devolve into the zeta function for the squared Dirac operator and the asymmetry

function respectively:

$$\zeta_{\mathcal{P}^2}(s) = \sum_{k,n,\kappa} (k+1) \left(\frac{\mu^{2s}}{[\lambda_{kn}^\kappa(B)]^{2s}} - \frac{\mu^{2s}}{[\lambda_{kn}^\kappa(0)]^{2s}} \right) \quad (11)$$

$$\begin{aligned} \eta(s) &= \mu^s \sum_{k,n,\kappa} (k+1) \left(\frac{\text{sgn}(\lambda_{kn}^\kappa(B))}{|\lambda_{kn}^\kappa(B)|^s} \right. \\ &\quad \left. - \frac{\text{sgn}(\lambda_{kn}^\kappa(0))}{|\lambda_{kn}^\kappa(0)|^s} \right). \end{aligned} \quad (12)$$

It should be stressed that in the presence of bag-like boundary conditions ζ_S and ζ_{AS} do not have the meaning of parity conserving and parity violating terms since a parity transformation in terms of eigenvalues is given by $\lambda(\vartheta) \rightarrow -\lambda(-\vartheta)$ and both spectral functions contain parity conserving and violating terms. Thus the determinant for a given parameter ϑ is defined by

$$\zeta'(0) = \left(\frac{1}{2} \zeta'_{\mathcal{P}^2}(0) \pm i \frac{\pi}{2} \zeta_{\mathcal{P}^2}(0) \mp i \frac{\pi}{2} \eta(0) \right), \quad (13)$$

with the normalization chosen in Eq.(9) such that $\zeta'(0)$ vanishes as $B \rightarrow 0$.

Spectral sums can be computed using a representation of the sum as a contour integral of the logarithmic derivative of the function whose zeroes determine the spectrum,

$$\sum_\lambda \frac{1}{\lambda^s} = \frac{1}{2\pi i} \oint_\Gamma \frac{d\xi}{\xi^s} \frac{d}{d\xi} \ln f(\xi), \quad (14)$$

see for example [10], where the zeroes of $f(\xi) = 0$ are $\xi = \lambda$ and the contour is chosen such that all zeroes are enclosed. With real parameter ϑ , the poles lie on the real axis, and there is no pole at the origin for any ϑ . By deforming the contour and accounting for the vanishing of contributions to the integral at infinity, the expressions arising from Eq. (14) can be transformed into real integrals. The following representations for the two spectral functions are eventually obtained

$$\begin{aligned} \zeta_{\mathcal{P}^2}(s) &= \rho^{2s} \frac{\sin(\pi s)}{\pi} \sum_{k=1}^{\infty} k^{1-2s} \\ &\quad \times \int_0^\infty \frac{dt}{t^{2s}} \frac{d}{dt} \Psi(k, t|z, \vartheta), \end{aligned} \quad (15)$$

$$\begin{aligned} \eta(s) &= \rho^s \frac{\cos(\pi s/2)}{i\pi} \sum_{k=1}^{\infty} k^{1-s} \\ &\quad \times \int_0^\infty \frac{dt}{t^s} \frac{d}{dt} \Phi(k, t|z, \vartheta), \end{aligned} \quad (16)$$

with Ψ and Φ being the sum of contributions from the two polarisations, taking the form

$$\begin{aligned}\Psi(k, t|z, \vartheta) &= \sum_{\kappa=\pm} \ln \left(\frac{A_{\kappa}^2(k, t|z) + e^{2\vartheta} B_{\kappa}^2(k, t|z)}{A^2(k, t) + e^{2\vartheta} B^2(k, t)} \right) \\ \Phi(k, t|z, \vartheta) &= \sum_{\kappa=\pm} \ln \left(\frac{A_{\kappa}(k, t|z) + ie^{\vartheta} B_{\kappa}(k, t|z)}{A_{\kappa}(k, t|z) - ie^{\vartheta} B_{\kappa}(k, t|z)} \right)\end{aligned}$$

and where

$$\begin{aligned}A_{-}(k, t|z) &= M\left(\frac{k^2 t^2 \rho^2}{4z}, k+1, z\right), \\ B_{-}(k, t|z) &= \frac{kt\rho}{2(k+1)} M\left(1 + \frac{k^2 t^2 \rho^2}{4z}, k+2, z\right), \\ A_{+}(k, t|z) &= M\left(-\frac{k^2 t^2 \rho^2}{4z}, k+1, -z\right), \\ B_{+}(k, t|z) &= \frac{2z}{kt\rho} \left[M\left(-\frac{k^2 t^2 \rho^2}{4z}, k+1, -z\right) \right. \\ &\quad \left. - \frac{k+1 + \frac{k^2 t^2 \rho^2}{4z}}{k+1} \right. \\ &\quad \left. \times M\left(-\frac{k^2 t^2 \rho^2}{4z}, k+2, -z\right) \right], \\ A(k, t) &= \frac{2^k k!}{(kt\rho)^k} I_k(kt\rho), \\ B(k, t) &= \frac{2^k k!}{(kt\rho)^k} I_{k+1}(kt\rho).\end{aligned}\quad (17)$$

The Bessel functions I_k emerge from the limit $B \rightarrow 0$ with the normalization of the determinant as specified above. Analytical continuation of $\zeta_{p^2}(s)$ and $\eta(s)$ can be done by means of standard methods (for an extensive review see [18]). These functions can be related to functions such as the Riemann zeta function $\zeta_R(s)$ with known analytical properties. Our particular example of such a calculation is clearly technically quite complicated. First of all, expansions of the confluent hypergeometric functions in $1/k$, similar to the Debye expansion of Bessel functions, is required,

$$M\left(\frac{k^2 t^2 \rho^2}{4z}, k+1, z\right) = C(t\rho, k) \sum_{n=0}^{\infty} \frac{M_n(t\rho, z)}{k^n}. \quad (18)$$

The form of the prefactors and the $M_n(x, z)$ functions will be presented in detail elsewhere [19]. It suffices to say that the prefactor $C(x, k)$ cancels out of Φ and Ψ after which the procedure is straightforward if somewhat tedious: the expansions Eq. (18) are inserted in $\zeta(s)$ and $\eta(s)$ so that the integrals over t can be evaluated term by term. After exchange of order of summation over k and n (which is the most subtle step here), sums over k can be read off in terms of the

Riemann zeta function. The resulting $\zeta_{p^2}(s)$ has the structure

$$\zeta_{p^2}(s) = s\rho^{2s} \frac{\sin(\pi s)}{i\pi} \sum_{n=0}^{\infty} \zeta_R(2s+n-1) f(z^2|n) + \delta\zeta_{p^2}(s), \quad (19)$$

where the term in the second line denotes those contributions coming from interchange of the order of summations over n and k , which are potentially present. A similar structure appears for the asymmetry spectral function as well. Finally we are interested in the decomposition of this expression around $s = 0$. The result for the first line in Eq.(19) comes from the $n = 2$ term alone and can be calculated with relative ease since only the lowest coefficients M_1 and M_2 in Eq. (18) contribute. However, as occurs in even simple problems [18], the second term in Eq.(19) is much more difficult to compute. To achieve this one needs to know coefficients $f(z^2|n)$ as a function of continuous variable n . Below we will concentrate on the contribution given in the first line alone, bearing in mind the necessity of a complete analysis. The final results for the first term in Eq. (19) and its analogue in the asymmetry function $\eta(s)$ are then summarised in the following equations:

$$\begin{aligned}\zeta_{p^2}(0) &= 0, \\ \zeta'_{p^2}(0) &= z^2 \left[-\frac{1}{4} - \ln 2 + \ln(1 + e^{2\vartheta}) \right] \\ \eta(0) &= \frac{z^2}{2} + \frac{z^2}{\pi} \text{Arctan}(\sinh(\vartheta))\end{aligned}\quad (20)$$

and hence

$$\begin{aligned}\zeta'(0) &= \frac{z^2}{2} \left[-\frac{1}{4} - \ln 2 \pm i\frac{\pi}{2} \right. \\ &\quad \left. + \ln(1 + e^{2\vartheta}) \mp i\text{Arctan}(\sinh(\vartheta)) \right] + \delta\zeta'(0),\end{aligned}\quad (21)$$

where $\delta\zeta'(0)$ denotes contributions potentially coming from the above-discussed non-commutativity of summations, which have been not calculated here. For an anti-self-dual background field, we merely have to perform a parity transformation on the fields. For the spectrum it involves $\lambda(\vartheta) \rightarrow -\lambda(-\vartheta)$. For the determinant, one need only take $\vartheta \rightarrow -\vartheta$ and $\eta(s) \rightarrow -\eta(s)$ independently.

4 Discussion and Conclusions

For interpretation of the result Eq. (21) one needs to continue ϑ to imaginary values. The appropriate continuation is $\vartheta \rightarrow i\alpha + i\pi/2$. In this case, one can

cleanly separate parity odd and even parts since for the self-dual field one gets

$$\zeta'(0) \rightarrow \frac{z^2}{2} \left[-\frac{1}{4} + \ln(1 \pm \cos(\alpha)) + i(\alpha + \pi\nu) \right],$$

with $\nu \in \mathbb{Z}$, while the anti-self-dual field leads to

$$\zeta'(0) \rightarrow \frac{z^2}{2} \left[-\frac{1}{4} + \ln(1 \pm \cos(\alpha)) - i(\alpha + \pi\nu) \right].$$

The imaginary part is given in a form which manifests the 2π periodicity in α . Recalling that the absolute value of the topological charge associated with the gluon field in a domain is $q = z^2/4$, the parity odd part becomes

$$\pm 2iq(\alpha \bmod \pi)$$

with the sign correlated with the duality of the field, and can be regarded as the axial anomaly. This agrees with the estimation of [1] obtained in a more general approach, up to a term $\pm iq\pi$. As far as we can see this difference occurs due to contributions in our calculation from both $\zeta'_{p^2}(0)$ and the asymmetry spectral function $\eta(0)$, which provides an extra $\pm iq\pi$ to give the above anomaly term. The contribution of the asymmetry spectral function was not considered in [1].

Final conclusions about the validity of the chiral angle dependence of the parity even part, as well as an analysis of the strong background field limit, necessarily require computation of the potentially nonzero term $\delta\zeta_p$ in Eq.(19).

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